International Journal of Marketing and Technology
Vol. 16 Issue 08, Aug 2026
ISSN: 2249-1058 Impact Factor: 8.559
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gate as well as in Cabell's Directories of Publishing Opportunities, U.S.A

ON COMPACT ANISOTROPIC SPHERES WITH SPECIFIED ENERGY DENSITY

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Abstract:

In this paper our objective is to generate a new class of exact solutions to Einstein field equation corresponding to a physically reasonable form for the density. A particular motive is to find simple analytic forms for the gravitational and matter variables so that the physical interpretation of the model is simplified. Often the solution are presented in terms of special functions or numerical approach is required [8, 9], we hope that our results in terms of elementary function will assist in the analysis of gravitational behavior of compact objects, and the study of anisotropy under strong gravitational fields. The relevant field equations are solved using a particular form for the energy density.

Key word : Exact, Anisotropic spheres, Density, Spherically symmetric, Gravitational fields.

Introduction :

Solutions of Einstein's field equations for spherically symmetric matter distribution have been solved by Adler [1], Whitman [25] Yadav et al. [26] and some other using different methods and assumptions. The matter distribution is usually assumed to be locally isotropic, in case of pressure for massive objects in general relativity. However, in the last few years theoretical studies on realistic steller models indicate that some massive objects may be locally anisotropic [2, 3, 4, 11, 18]. There are a number of interesting solutions that have provided insight into the effects of anisotropy on star parameters [6, 13, 17]. However, many of these solutions have a limited applicability to astrophysical situations. Since they do not satisfy certain physical restrictions usually imposed upon density and pressure, viz., that the pressure should not exceed the energy density (dominant energy condition), and that the (adiabatic) derivatives of the pressure with respect to the density should be less than or equal to unity [12] (macro causality conditions).

In recent years a number of authors have studied solutions to the Einstein field equations corresponding to anisotropic matter where the radial component of the pressure differs from the angular component. The gravitation field is taken to be spherically symmetric and static since these solutions may be applied to relativistic stars. A number of researchers have examined how anisotropic matter affects the critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the papers [5, 6, 7, 10, 14, 15, 16, 19, 24] among others. Some researchers have suggested that anisotropy may be important in understanding the gravitational behavior of boson stars and the role of strange matter with densities higher than neutron stars, Mak and Harko [21, 22] and Sharma and Mukherjee [23] suggest that anisotropy is a crucial ingredient in the description of dense stars with strange matter.

2. The Field Equations and Their Solution

The line element for static spherically symmetric space times is give by

$$ds^2 = e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta df^2) \quad (2.1)$$

v(r) and (r) are arbitrary functions. We study non-radiating relativistic model with anisotropic stress and the energy momentum tensor is of the form.

$$T^{ab} = \mu u^a u^b + p h^{ab} + \pi^{ab} \quad (2.2)$$

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where μ is the energy density, p the isotropic pressure and the projection tensor $h^a = u^a u^b + g^{ab}$ is measured relative to the four-velocity u^a. It is convenient to express the anisotropc stress in the form.

(2.3) $$\lambda^{ab} = \sqrt{3}\, S(r)\left(C^a c^b - \frac{1}{3} h^{ab} \right)$$

where the unit spacelike vector c is orthogonal to the fluid four-velocity u and |s(r)| is the magnitude of the stress tensor. This representation for π^{ab} is a consemence of the symmetries of the static spherically symmetric spacetimes [13]. The quantity S is a useful device to introduce.

$$p_r = p + 2S/\sqrt{3}, p\perp = p/\sqrt{3}$$

which are the radial and tangential pressures respectively. Note that for isotropic mater S = 0 and $p_r = p\perp = p$. The magnitude S provides a measure of anisotropy.

We assume that the fluid four-velocity is commoving. This assumption implies that $u^a = e^{-\nu/2}\delta^a, C^a = e^{-\lambda/2}\delta^2$

Using (2.1) and (2.2) the Einstein field equation become.

(2.3a) $$-\frac{e^{-\lambda}}{r^2}(1 - \lambda r - \theta) = \mu$$

(2.3b) $$\frac{e^{\lambda}}{r^2}\left(1 - e^{\lambda} + r\nu\right) = pr$$

(2.3c) $$\frac{1}{r^2}\left(2\nu - \nu'\lambda' + \bar{\nu}^2 + \frac{2\nu'}{r} - \frac{2\lambda'}{r} \right) = p\perp$$

for static spherically symmetric anisotropic matter. We are using units where the speed of light and the coupling constant are unity. The momentum conservation equation leads to.

(2.4) $$(\mu + p_r)\nu' + 2p_r' + \frac{1}{r}(p_r - r)_{\perp} = 0$$

For the spacetime (2.1). this conservation equation is not independent and can be generated directly from the field equation (2.3), We define the mass function as

(2.5) $$m(r) = \frac{1}{2}\int_0^r x^2 \mu(x) dx$$

following the treatment of Stephani [24]. With the help of (2.4) and (2.5) we can integrate (3a), and then get the equivalents system

(2.6a) $$e^{-\lambda} = 1 - \frac{2m}{r}$$

(2.6b) $$r(r - 2m)\nu' = p_r r^3 + 2m$$

(2.6c) $$(\mu + p_r)\nu' + 2p_r' = \frac{4}{r}(p_r - p)_{\perp}$$

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The system has the advantage of being a first order system of differential equations, and is linear in the gravitational potential v which simplifies the integration process. For certain application it is easier to use rather than the original second order system, which is the approach that we follow in this paper. We seek explicit solutions to the Einstein field equation that describe reaslistic anisotropic relativistic stars by utilizing an algorithm that we initially proposed by Maharaj and Maartens [20]. In their approach they expressed the field equation as the first order system of differential equation. The energy density (or equivalently m) and the radi al pressure re p_r are chose on physical grounds. The remining relevant quantities $\left(e^{\lambda}, e'', p\right)$ then follow from the field equations. Note that (ev, eλ, μ(or m) p_r, p) are not independent, there are five unknown functions and three field equations so that we have the freedom to choose any two of the quantities. In this paper we make explicit choices for μ and p_r.

It is convenient to make the following choice for the energy density,

$$\mu = \frac{j}{r^2} + k + \ell r^2 \tag{2.7}$$

where j, k and l are constants.

The roles of j, k and l in the physics of the model are highlighted in examples considered later. An advantage of this form for μ is that it contains particular cases studied previously. They (2.5) yields the following expression for the mass function.

$$m = \frac{r}{2}\left(j + \frac{k}{3}r^2 + \frac{l}{5}r^4 \right) \tag{2.8}$$

with the particular energy density (2.7) Equation (2.6a) gives

$$e^{-\lambda} = 1 - j - \frac{k}{3}r^2 - \frac{l}{5}r^4 \tag{2.9}$$

and the gravitational potential has been determined. With the help of (2.8) we can write (2.6b) as

$$\nu' = \frac{rp_r}{1 - j - \frac{k}{3}r^2 - \frac{l}{5}r^4} + \frac{j + \frac{k}{3}r^2 + \frac{l}{5}r^4}{r(1 - j - \frac{k}{3}r^2 - \frac{l}{3}r^2 - \frac{l}{3}r^4)} \tag{2.10}$$

$$= \frac{rp_r}{1 - j - \frac{k}{3}r^2 - \frac{1}{5}r^4} + \frac{j}{r(1-j)}$$

$$+ \frac{\frac{k}{3}r + \frac{l}{5}r^3}{(1-j)(1 - j - \frac{k}{3}r^2 - \frac{1}{3}r^4)}$$

Where we have used partial fractions. On integration it can be expressed as

$$\nu = I_1 + \frac{j}{1-j}\ln r + \frac{1}{1-j}I_2 + \ln B \tag{2.11}$$

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Where in B is a constant of integration and we have set.

$$I_1 = \int \frac{rp_r}{1 - j - \frac{k}{3}r^2 - \frac{1}{5}r^4} dr$$

$$I_2 = \int \frac{\frac{k}{3}r + \frac{1}{5}r^3}{1 - j - \frac{k}{3}r^2 - \frac{1}{5}r^4} dr$$

At this point we could choose a barotropic equation of state $p_r = p_r(\mu)$. However this is an approach that we intend to follow in future work. In this treatment we make a choice for the radial pressure p_r which is physically reasonable and is a generalization of earlier studies. We make the choice.

(2.12) $$p_r = \frac{C}{1-j}\left(1 - j\frac{k}{3}r^2 - \frac{1}{5}r^4\right)\left(1 - \frac{r_2}{R^2}\right)^n$$

When $j = \ell = 0$, we obtain the radial pressure postulated by Maharaj and martens [20]. For $j = 0$, we regain the radial pressure of Gokhroo and Mehra [10]. The form (2.12) for p_r is physically reasonable because $p_r > 0$ in the interval (0, R)for relevant choices of the constant, $p_r = C$ at the centre $r = 0$, $p_r = 0$ at the boundary $r = R$ and p_r is continuous and well behaved in the interval [0, R].

The first integral l_1 simplifies to

$$l_1 = \frac{c}{1-j}\int\left(1 - \frac{r^2}{R^2}\right) rdr$$

$$= -\frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}$$

For the choice of (2.12), to evaluate the second integral l_2 we need to consider two case : 1 = 0 and l ≠ 0.

Case I : $l = 0$

In this case the integration is straight forward and we obtain f

$$l_2 = -\frac{1}{2}\ln\left\{1 - j - \frac{k}{3}r^2\right\}$$

Case II : $\ell \neq 0$

For this case we et

$$u = r^2 + \frac{5k}{6\ell_2}, q^2 = 1 - j + \frac{5k^2}{36\ell}$$

and obtain

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$$l_1 = \int \frac{l_r\left(r^2+\frac{5k}{3\ell}\right)}{1-j+\frac{5k^2}{36\ell}-\frac{\ell}{5}\left(r^2+\frac{5k}{6\ell}\right)^2}dr$$

$$= \frac{\ell}{10}\int \frac{u+\frac{5k}{6\ell}}{q^2-\frac{\ell}{5}u^2}$$

$$= \frac{\ell}{10}\left(-\frac{5}{2\ell}\ln\left\{q^2-\frac{\ell}{5}u^2\right\}+\frac{5k}{6\ell}\left(\frac{\sqrt{5}}{q\sqrt{\ell}}\right)\tanh^{-1}\left\{\frac{u\sqrt{\ell}}{q\sqrt{5}}\right\}\right)$$

Hence we can collectively write for both Case I and Case II that

(2.13)
$$\begin{bmatrix} -\frac{1}{2}\ln\left\{1-j-\frac{k}{3}r^2\right\}, \text{ for } \ell=0 \\ -\frac{1}{4}\ln\left\{1-j+\frac{5k^2}{36\ell}-\frac{\ell}{5}\left(r^2+\frac{5k}{6\ell}\right)^2\right\} \\ +\left(\frac{5}{\ell}\right)^{\frac{1}{2}}\left(\frac{k}{\sqrt[12]{1-j+\frac{5k^2}{36\ell}}}\right)\tanh^{-1}\left(\frac{\ell}{5}\right)^{1/2}\frac{r^2+\frac{5k}{6\ell}}{\sqrt{1-j+\frac{5k^2}{36\ell}}} \end{bmatrix}$$

Integral l_1 and l_2 are given in terms of elementary functions which helps in physical analysis of the model.

On substituting l_1 in (2.11) we obtain.

(2.14) $$e^{\nu} = Br^{1-\frac{i}{j}}\exp\left\{\frac{l_2}{1-j}-\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}$$

The gravitational potential e^{ν} where $l_2 = \underline{l}_2(r)$ has the functional representation given above in for $t \neq 0$. To match the interior solution the Schwarzschild interior we require the $e^{\nu(R)}$ = 1-2 M/R which implies $AB = R^{\frac{-1}{1-1}}(1-2M/R)\exp\{-l_2(R)/(j-1).$ Finally the last field equation (2.6c)

(2.15) $$p_{\perp} = p_r + \frac{C}{2(1-j)}\left(j-\frac{\ell}{r}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n$$

$$+\frac{r^2}{2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^{-1}$$

$$\times\left\{\frac{C^2}{2(1-j)^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$$

$$-\frac{2nC}{(1-j)R^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)$$

$$+\frac{1}{r^2}\left(\frac{j}{r^2}+j+\ell r^2\right)\left(j+\frac{k}{5}r^2+\frac{\ell}{5}r^4\right)$$

where we have used (2.7), (2.12) and (2.14). the anisotropic factor S(r) is given by

(2.16) $$S=\frac{C}{2\sqrt{3}(1-j)}\left(j-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n$$

$$-\frac{r^2}{2\sqrt{3}}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)$$

$$\times\left\{\frac{C^2}{2(1-j)^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$$

$$-\frac{2nC}{(1-j)R^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}$$

$$\left.+\frac{1}{2r^2}\left(\frac{j}{r^2}+k+\ell r^2\right)\left(j+\frac{k}{3}r^2+\frac{\ell}{5}r^4\right)\right\}$$

Which follows from

Thus we have generated a new class of solutions to the Einstein field equation Collecting the various results given above we can express the exact solution as

(2.17a) $$\mu=\frac{j}{r^2}+k+\ell r^2$$

(2.17b) $$p_r=\frac{C}{1-j}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n$$

(2.17c) $$p_\perp=p_r+\frac{C}{2(1-j)}\left(j-\frac{\ell}{5}r^4\right)\left(1-\frac{r^2}{R^2}\right)^n+\frac{r^2}{2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^{-1}$$

$$\times\left\{\frac{C^2}{2(1-j)^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$$

$$-\frac{2nC}{(1-j)R^2}\left(1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}$$

$$\left.+\frac{1}{2r^2}\left(\frac{j}{r^2}+k+\ell r^2\right)\left(j+\frac{k}{3}r^2+\frac{\ell}{5}r^4\right)\right\}$$

(2.17d) $e^{\nu}=Br^{\frac{1}{1-j}\exp\left\{\frac{\ell_2}{1-j}-\frac{CR^2}{2(1-j)(n+1)}\left(1-\frac{r^2}{R^2}\right)^{n+1}\right\}}$

(2.17e) $$e^{\nu}=\frac{1}{1-j-\frac{k}{3}r^2-\frac{\ell}{5}r^4}$$

where l_2 (given in contains the two cases, $\ell\neq 0$. The exact solution represents the interior of an anisotropic star corresponding to the thergydensity $M=\frac{j}{r^2}+k+\ell r^2$ Clearlv other choices for μ and β will yield new solutions to the field equations, however these choices may not correspond to realistic matter or the integral l_1 and l_2 may not be expressible in closed form. This solution does not have a barotropic equation of state $p_r = p_r(\mu)$. To obtain a model with an equation of state we need to specify this explicitly when evaluating the integral l_1. An equation of state is a desirable physical feature which we hope to incorporate in future models. Note that the cosmological constant is absent from our model. This quantity can be easily included by adding a constant to the energy density and the pressure function.

The gravitational potential e^{λ} is finite at the centre r = 0 and at the boundary r = R. The function e^{λ} is well behaved in interior of the relativistic star. The gravitational potential e^{ν} is continuous and well behaved in the interior and finite at the boundary of the star r = R. There is a singularity at the centre r = 0 in the potential e^{ν} . The singularity in e^{ν} is removable for a specific choice of parameter values. This singularity is eliminated by setting j = 0 which corresponds to the solution of Gokhroo and Mehra [10].

3. References

1. Adler, R.J. (1974) J.M.P., 15, 727.
2. Binney, J., Tremaine, S. : Galactic Dynamics, Princeton University Press, Princeton. JN (1987), Phys. Rev., D 26, 1262.
3. Bondi, H. 2(a) Bayin, S.S. (1982) Phys. Rev. D26, 1262Mon. Not, R. Astr. Soc. 259, 365-368 (1992).
4. Bowers, r.L. Liang, E.P.T. (1974) Astrophys. J. 188, 657-665.
5. Buchdahal, H.A. (1959) (Phys., Rev. 116, 1027-1034.
6. Cosenzal, M. et. al. (1981); J.M.P., 22, 118.
7. Cuddeford, P. (1991) Mon. Not. R. Astr. Soc., 253, 414-426.
8. Dev, K., Gleiser, M. (2002) Gen., Rel. Grav. 34, 179-1818.

9. Dev, K., Gleiser. M. (2003) Gen. Rel. Grav. 35, 1435-1457.
10. Gokhroo, M.K., Mehra, A.L., (1994) Gen. Rel. Grav. 26, 75-84.
11. Hansraj. S. : Exact solution and conformal rormations in general relativity, Phy. D. thesis, University of natal (1999).
12. Hawking S.W. and Ellis, G.F.R. (1973). The large scale structure of space time, Cambridge UP.
13. Herrelra L and Leon J.P. de (1985), J.M.P. 26, 2018.
14. Herrera, L., Ruggeri, G.J., Witten. L., (1979) Astrophys. J. 234, 1094-1099.
15. Herrera, L. Santos, N.O. (1997) Phys. Rep. 286, 53-130.
16. Ivanov, B.V. (2002) Phys., Rev., Rev. D65, 10411.
17. Krori et al. (1983) : Can. J. Phys. 62, 239.
18. Letelier, P.S. (1980), Phys. Rev., D22, 807.
19. Maharaj. S.D., Maartens, R.J. (1986) Math. Phys. 27, 2514-2519.
20. Maharaj, S.D., Maartens, R. (1989) Gen. Rel. Grav. 21, 899-905.
21. Mak, M.K., Harko, T. : Chin. J. (2002) Astron. Astrophys. 2, 248-259.
22. Mak. M.K., Harko. T. (2003) Proc. Roy. So. Lond. A459, 393-408.
23. Sharma, R., Mukherjee, S. (2002) Mod., Phys. Lett. A. 17, 2535-2544.
24. Stephani, H. : Relativity : An Introduction to Special and General Relativity, Cambridge University Press (2004).
25. Whitman, P.G. (1977) : J.M.P. 18, 868.
26. Yadav, R.B.S. Sinha A.K. and Kumar, M. (2007)Acta Ciencia India, 33, 185.

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